1967-12

An analytical model for application to the operation of replenishment at sea

Waggoner, Mark Harvey

Monterey, California. U.S. Naval Postgraduate School

http://hdl.handle.net/10945/11640

AN ANALYTICAL MODEL FOR APPLICATION TO THE OPERATION OF REPLENISHMENT AT SEA

MARK HARVEY WAGGONER

82

AN ANALYTICAL MODEL FOR APPLICATION TO THE OPERATION OF REPLENISHMENT AT SEA

by

Mark Harvey Waggoner
Lieutenant, United States Navy
B.S., Naval Academy, 1961

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
December 1967

ABSTRACT

An analytical model for replenishment at sea is formulated for two supply ships and L combatants using queueing theory concepts and a random walk model in the plane. Exponential distributions are assumed for replenishment times, and, given the initial number of combatants to be replenished by each supply ship, the distribution for total time to complete the finite operation is obtained in terms of Laplace transforms. All possible sequences for finishing the replenishments of the combatants have been considered in the model, and the techniques which were developed to count the number of sequence possibilities are presented as an appendix. Although this model involves only two supply ships, it is believed that the methods used may be generalized for application to more complicated models.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ACKNOWLEDGEMENT

The writer greatly appreciates the invaluable assistance and guidance of Professor Paul R. Milch during the course of this investigation.

I. Introduction

The capability of the United States Navy to maintain fleets at sea, fully ready to carry out any assigned task, is an important asset in the ever increasing global responsibilities of this nation. To achieve this capability, the necessary logistic support for naval combatant forces is provided by replenishment at sea. This underway replenishment (UNREP), however, must be accomplished without interfering with the primary mission of the supported force. Consequently, the principle aim of underway replenishment is the safe delivery of a maximum amount of supplies in a minimum of time.

An UNREP is accomplished primarily by means of intership horizontal transfers via rigs connecting the supply ships and the supported units. Normally, there is a supported unit on each side of the supply vessel. This necessity for working at close quarters makes maneuvering a critical operation. Speed and course changes are restricted, and speed is necessarily slower than normal. The increased vulnerability of forces while replenishing and the increased hazards associated with close operations further enhance the importance of minimizing total replenishment time.

Extensive training, revised delivery techniques, and newer types of supply ships are among the ways the Navy is presently combatting the time problem. However, the overall efficiency and effectiveness of an UNREP is usually directly proportional to the thoroughness of prior planning. A detailed knowledge of the limitations, capabilities, and

requirements of all units involved in an UNREP is essential to properly schedule ships for a successful replenishment.

Aids for efficient planning and models for studying the effects of this planning are limited. McCullough (5) made an analytic approximation of the replenishment process by using a multi-stage cyclic-queueing model. The model considered M supply ships, the stages, placed in series. These ships serviced N combatant units, each of which passed by the supply ships in succession. An infinite queue with cycles was then assumed by allowing the combatant ships to repeat the operation indefinitely, and the long run (or "steady state") behavior was studied. The solution to this model provided an upper bound for a computer simulation also considered.

Although McCullough's model gives some insight into the UNREP process, it does not really represent the actual operation of replenishment at sea. An UNREP is not cyclic in nature, because the sequence of operation is not repeated. It is a finite operation with series queues in parallel. The number of combatant ships is fixed, and there are no new arrivals. Consequently, a time dependent (not "steady state") solution is required.

Gordon and Copes (2) developed a deterministic model for the planning of a replenishment operation by treating the UNREP as a special case of a job-scheduling problem. The service times of each ship were assumed to be known, and general expressions were obtained for the total time to complete the UNREP and the total waiting time of the ships

involved. A solution was derived which considered a maximum replenishment force of three supply ships and only three combatant ships, although the techniques developed were believed extendable to larger operations. Unfortunately, when service times are predetermined and fixed, as in this model, the possibility of a mishap or unexpected event occurring is not anticipated. Rig failures, broken lines, or accidents could change the replenishment time for a given ship. Thus, it seems reasonable to assume service times are random variables rather than deterministic.

II. Background

The general UNREP problem is concerned with a flow of customers requiring services, a trait common to all queueing systems. However, an UNREP differs from most queueing processes because it involves finite series queues which start simultaneously at every service facility. Each facility may provide a different type service, and every customer normally requires all services that are provided. The order for receiving services is predetermined for every customer, but each customer does not follow the same sequence, so that there is a different series queue for each customer. All queues are finite, and there are no new system arrivals. The operation is finished when the original number of customers in the system has received all required services.

There are many complications associated with the UNREP problem alone, but finite inputs to systems with waiting lines at several service facilities are not uncommon. A similar problem is evident at a garage when a given number of cars require the same services and order of service is unimportant, i.e., services such as tire rotations, oil changes, and gasoline fill-ups. Another example materializes in a commercial store at closing time, when a given number of customers all require services at several different counters. A group of refreshment stands catering to a given number of people also falls in this category.

The most practical examples are probably the multistage production processes. These processes have been studied in

the category of job-shop problems, and the similarity to an UNREP was indicated by Gordon and Copes (2). The known job-shop investigations to date, however, are concerned with the case in which all jobs must start with the first machine. Other constraints usually considered involve definite sequences of operations and time limitations for each job.

Bellman (1) has described a number of simple prototype multistage problems and touched on some of the analytical and computational techniques used in early investigations. Many of the complications involved in an UNREP were considered, but never were all present in the same model.

Sisson (6) defined the job-shop process and reviewed several methods for sequencing in job shops. Two basic models for the job-shop sequencing problem were discussed, but they were presented only as an aid for an intuitive understanding of the situation. A complete solution was not found.

In a more recent work, Smith and Dudek (7) describe an algorithm that yields an optimal sequence for n-jobs requiring processing through M-machines when no passing is allowed. A pre-scheduled sequence is assumed, times are deterministic, all jobs commence with the first machine, and only one series is considered, therby severely restricting its use in an UNREP model.

Service facilities in series have also been investigated using queueing theory, but, although there have been numerous studies in recent years, the models presented cannot be simply modified to represent the UNREP situation. Only a

relatively few of the studies have even considered the restriction of finite queues, and these studies usually do not limit the queue size in front of the first server. One of the first investigations of finite queues under these conditions was performed by Hunt (4), who derived the maximum possible utilization for four particular cases of service facilities in series: infinite storage between stages, no storage between stages, finite storage space between stages, and the case of the unpaced belt-production line. The corresponding expected number of customers in the system under the assumption of exponential service times was also obtained.

Hillier and Boling (3) extended Hunt's work in terms of numerical results and numerical procedures that made it possible to analyze larger systems having exponential or Erlang service times. But, the input process considered was again such that the first queue was never empty.

Allowing only finite inputs, starting all service facilities at the same time, simultaneously allowing different sequences of machine usage or service, and keeping all facilities occupied, as desired in an UNREP, create difficult twists to the job-shop problem or series queue situation. Unfortunately, there are no published works, to the author's knowledge, which have considered problems of this nature.

III. Description of a Replenishment Operation

In an actual underway replenishment, there can be several different types of supply ships (servers) and different types of combatants (customers). A complication that is immediately evident is that the quantity of each type of supply ship (and/or the combatant) can be different. For instance, a typical underway replenishment group could consist of three AOs (oilers), one AK (cargo ship), and two AEs (ammunition ships) replenishing five DDs (destroyers) and one CVA (attack carrier), or one AO and one AE replenishing a CVA, a CAG (guided missile cruiser), and four DDs. This latter composition is typical of the frequent UNREPs in the South China Sea.

Each supply ship is capable of servicing two ships simultaneously (excluding helicopter operations). The service rate is normally different for each type of server and also varies according to the type of combatant being serviced. The combatant order for replenishing is predesignated so that, technically, lines are formed behind each server when the operation commences, but the service sequence is not the same for all customers. Each combatant replenishes at one supply ship and then proceeds to another line, replenishing from each type of server only once. The operation ends when all combatants have been replenished.

The many complications of the UNREP problem make it amenable for computer simulation, but computer programs can be expensive to run, and computers are not always available to the planning staffs. Therefore, it seems desirable to

develop an analytical model which could realistically approximate an underway replenishment operation. The model described in the following pages is presented as a first approach to the problem.

IV. Formulation of the Problem

Although a model for the general case of m servers and n customers is desirable, for simplicity, attention is restricted to the case of two different servers and a given number of customers. Each customer is serviced by both servers. The distribution of time for completion of an UNREP is obtained. Having obtained this solution, it is believed that the problem may be generalized to include a larger number of supply ships.

It should be noted that the solution to this problem is still a practical one. Many UNREPs presently conducted in the South China Sea involve one or two supply ships, each replenishing any number of ships of the same type.

The force composition being investigated is two supply ships, A and B, and L combatants of the same type. Each supply ship replenishes only one combatant at a time. When the UNREP begins, the L combatants are divided into two service lines, M ships waiting for server A and $N = L - M$ ships in B's service line. No additional ships join the queues once the operation has started.

The order of service is predetermined. When a combatant has been serviced by A, that ship joins the queue behind B. Likewise, a combatant proceeds to A's line when replenishment is completed from B. The operation is finished when all combatants have been serviced by both supply ships. Figure 1 indicates the initial queues and the flow from one line to the other.

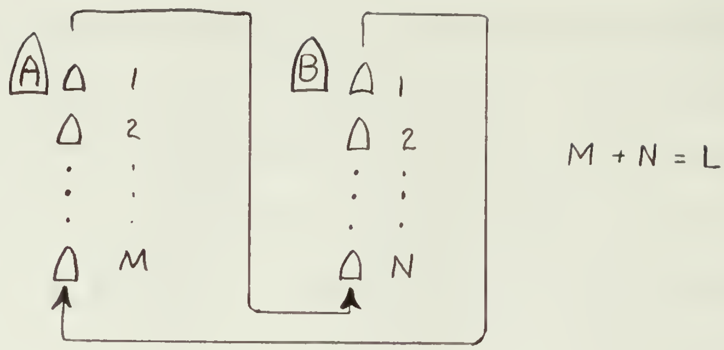

Figure 1

Each server acts independently of the other server, and customer services are independent, so that all service times are independent. Transit times between supply ships are assumed identical for all customers and considered negligible in comparison with service times. The two servers have service times exponentially distributed, with service rates of λ and μ, respectively. It is hoped that this restriction may be reduced in future studies. The exponential distribution frequently does fit many realistic queueing situations. However, to insure a more accurate model, the service time distributions should be determined from known operational data. This is a problem for further study and is not considered here.

V. The Model

A. Graphical Representation

Given L combatants to be replenished by both of two supply ships, A and B, the distribution for the total UNREP time can be determined with the aid of two variables, U_t and V_t. U_t represents the number of combatants that A replenishes by time t, and V_t denotes the number that B replenishes by time t.

Starting at the origin of a graph, a horizontal step of unit length will be made to the right if A finishes replenishing a combatant before B; if B finishes first, a vertical step of unit length will be made upwards. The second, and subsequent, steps will also be to the right or up, depending on whether the next customer finishing is serviced by A or B. With the abscissa of the graph as U_t and the ordinate as V_t, the coordinates (U_t, V_t) indicate the state of the UNREP at

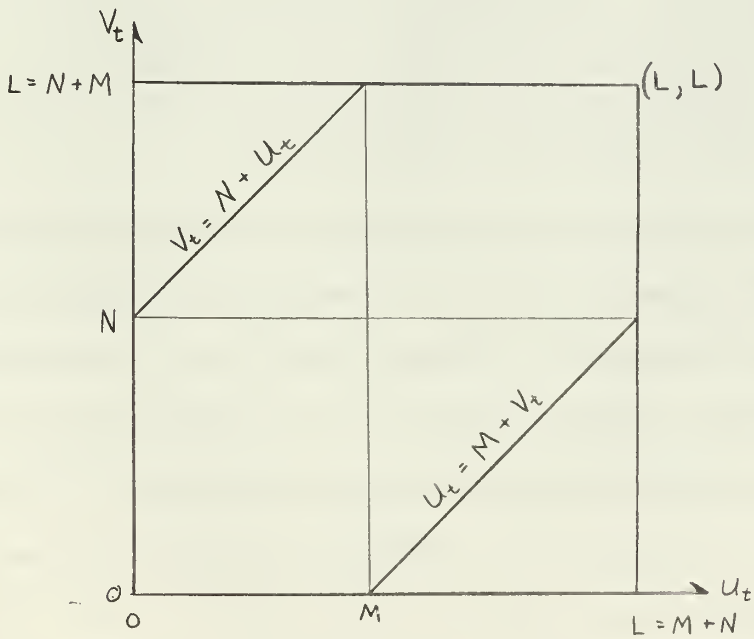

Figure 2

any time t, and the UNREP process may now be regarded as a random walk in this plane.

A specific sequence of steps from the origin to (L,L) will represent an entire UNREP process. However, not all paths are possible. If A has replenished the M combatants initially waiting in his line, $U_t = M$, and B has not completed any replenishments, $V_t = 0$, then A will be idle until B has replenished one ship. Likewise, if $U_t = M + 1$ and $V_t = 1$, A will again be idle until $V_t = 2$. This leads to the restriction that $U_t \leqq M + V_t$, and, since the total number of replenishments by A is L, $U_t \leqq M + V_t \leqq L$. Similarly, $V_t \leqq N + U_t \leqq L$. Figure 2 depicts the graph with restrictive boundaries.

B. Distribution Development

Now examine a specific UNREP sequence from start to finish. Let X represent the service time of A and Y the service time of B. By definition, let

$$P(X \leqq t) = 1 - e^{-\lambda t}, \; t \geqq 0 \quad \text{and}$$

$$P(Y \leqq t) = 1 - e^{-\mu t}, \; t \geqq 0 \; .$$

When the process begins, the first step on the graph in the UNREP sequence will be either horizontally to the right if A's service time, X, is less than B's, Y, or vertically up if the situation is reversed, Y is less than X. It is shown in appendix A that the conditional distribution of service time for A, given X is less than Y, is again exponential, but with parameter $\lambda + \mu$, i.e.,

$$P(X \leqq t | X < Y) = 1 - e^{-(\lambda + \mu)t}, \; t \geqq 0 \; .$$

Likewise, the conditional service time distribution for B, given Y is less than X, is also exponential with parameter $\lambda + \mu$. Therefore, for the first step,

$$P(Y \leq t | Y < X) = P(X \leq t | X < Y) = 1 - e^{-(\lambda + \mu)t}, \; t \geq 0 .$$

The second step of the sequence will be examined from the point $(U_t = 1, V_t = 0)$. Again, the choice for this step is either to the right or up, depending on whether A or B finishes first. Since A is servicing a new customer, the distribution of service time for A is known. B is still replenishing his first customer. However, by employing the memoriless property, $P(X \leq T | X < t) = P(X \leq T-t)$ when $t < T$, of the exponential, the distribution of the service time that is left for B remains exponential with rate μ. Therefore, the problem is exactly as before, and the conditional distribution of time for the second step is again exponential with parameter $\lambda + \mu$, given that it is a vertical, or given that it is a horizontal, step.

In general, this conditional distribution is true at any point except those points on the boundary, i.e., when $U_t = M + V_t \leq L$ or $V_t = N + U_t \leq L$. If $U_t = M + V_t \leq L$, then the path position for the process is on the right boundary, and B is working while A is idle. Therefore, the next step must be up, and the distribution of time is the same as the distribution for Y, exponential with parameter μ. This is again due to the memoriless property of exponential distributions. Similarly, when $V_t = N + U_t \leq L$, the location is on the upper boundary, and the next step must be to the right with the same distribution of time as X.

Let E_{ij} denote the event that a given route has i positions on the upper boundary and j positions on the right boundary. Then, when E_{ij} occurs, 2L - (i + j) positions along the path are not on a boundary.

The total time T for a given UNREP sequence is, therefore, just the sum of the times between each of the positions on the path. Let Z be the random variable with $P(Z \leq t) = 1 - e^{-(\lambda + \mu)t}$; then

$$T = Z_{2L-i-j} + X_i + Y_j, \quad \text{if event } E_{ij} \text{ occurs.}$$

The subscripts on Z, X, and Y indicate the number of times each variable is summed, i.e., X_i is the sum of i exponential random variables, each with parameter λ.

C. Total Time Probability Statement

Unfortunately, a given E_{ij} can occur in many ways, and, in addition, there are many combinations of i and j that are also possible. However, once the probability of event E_{ij} is determined, the unconditional distribution of UNREP time can be ascertained by applying the theorem of total probability. The unconditional probability statement is, therefore:

$$P(T \leq t) = \sum_{i,j} P(T \leq t | E_{ij}) \cdot P(E_{ij}) .$$

VI. The Solution for Total Time

A. Path Probabilities

Consider once more the step sequence representing an UNREP process. Starting at the origin, the first step in the sequence will be to the right with probability p, or up with probability 1 - p = q. It can be shown (See appendix A.) that $p = \frac{\lambda}{\lambda + \mu}$ and $q = \frac{\mu}{\lambda + \mu}$. Then, considering any point along the route except those points on the boundaries, and employing the memoriless property of the exponential distribution, the probability that a step will be to the right is p and that a step will be up is q.

When a position is on the right boundary, the next step must be up with probability one. Similarly, from positions on the upper boundary, the next step must be to the right with probability one. Consequently, when E_{ij} occurs, i steps to the right and j steps up are made at the boundaries. Since L steps are made in each direction, L - i steps are to the right, each with probability p, and L - j vertical steps are made, each with probability q.

For a given E_{ij}, however, several different step sequences are obviously possible. Therefore, for specific i and j, the number of possible routes must be counted before the probability of E_{ij} can be determined. Utilizing combinatorial analysis and the reflection principle, a counting technique is developed in Appendix B which yields R_{ij}, defined as the number of possible ways in which E_{ij} can occur. The results are as follows [Note: $C(n,r) = n!/r!(n-r)!$]:

1. $R_{00} = 0$

2. $R_{i0} = C(2L-i-1,\ L-1) - C(2L-i-1,\ M-i) - C(2L-i-1,\ N-i),$

   $i = 1,2,\ldots,L$

3. $R_{0j} = C(2L-j-1,\ L-1) - C(2L-j-1,\ M-j) - C(2L-j-1,\ N-j),$

   $j = 1,2,\ldots,L$

4. $R_{ij} = C(2L-k-1, M-k+1) + C(2L-k-1, N-k+1) - C(2L-k-1,\ M-k) -$ $C(2L-k-1,\ N-k)$, where $k = i + j = 2,3,\ldots,$ $\max\ (M,N) + 1$, and $i \geqq 1,\ j \geqq 1$.

Knowing R_{ij}, the probability of E_{ij} is determined below:

$$P(E_{ij}) = R_{ij}p^{L-i}q^{L-j} \quad .$$

B. Transform of the Distribution

The necessary equations and expressions for determining the total time distribution are now known and are summarized briefly below:

1. $T = Z_{2L-i-j} + X_i + Y_j$, given E_{ij},

2. $P(E_{ij}) = R_{ij}p^{L-i}q^{L-j}$,

3. $P(T \leqq t) = \sum_{i,j} P(E_{ij}) \cdot P(T \leqq t|E_{ij})$.

Since the random variables X_i, Y_j, and Z_{2L-i-j} are mutually independent, the Laplace transform, denoted $f^*(s)$, will be used to determine the total time distribution.

$$f_T^*(s) = E(e^{-sT}) = \sum_{i,j} [P(E_{ij})] f_{X_i}^*(s) \cdot f_{Y_j}^*(s) \cdot f_{Z_{2L-i-j}}^*(s)$$

$$= \sum_{i,j} [P(E_{ij})][f_X^*(s)]^i [f_Y^*(s)]^j [f_Z^*(s)]^{2L-i-j}$$

$$= \sum_{i=0}^{L} \sum_{j=0}^{L} R_{ij} \left(\frac{\lambda}{\lambda+\mu}\right)^{L-i} \left(\frac{\mu}{\lambda+\mu}\right)^{L-j} \left(\frac{\lambda}{\lambda+s}\right)^{i} \cdot \left(\frac{\mu}{\lambda+s}\right)^{j} \left(\frac{\lambda+\mu}{\lambda+\mu+s}\right)^{2L-i-j}$$

$$= \frac{\lambda^L \mu^L}{(\lambda+\mu+s)^{2L}} \sum_{i=0}^{L} \sum_{j=0}^{L} R_{ij} \frac{(\lambda+\mu+s)^{i+j}}{(\lambda+s)^i (\mu+s)^j}$$

Let $\alpha = \frac{\lambda+s}{\lambda+\mu+s}$ and $\beta = \frac{\mu+s}{\lambda+\mu+s}$. Then substituting the values of R_{ij} and evaluating, if $\lambda \neq \mu$:

$$f_T^*(s) = \frac{\lambda^L \mu^L}{(\lambda+\mu+s)^{2L}} \cdot \Bigg[\sum_{j=1}^{L} C(2L-j-1,\ L-1)\left[\alpha^{-j} + \beta^{-j}\right]$$

$$- \sum_{j=1}^{N} \Bigg\{ C(2L-j-1,\ N-j)\left[\alpha^{-j}\left(\frac{\mu+s}{\mu-\lambda}\right) + \lambda^{-j}\left(1 - \frac{(\mu+s)^2}{(\lambda+s)(\mu-\lambda)}\right)\right]$$

$$- C(2L-j-2,\ N-j)\left[\alpha^{-j-1}\left(\frac{\lambda+s}{\mu-\lambda}\right) - \beta^{-j-1}\left(\frac{\mu+s}{\mu-\lambda}\right)\right]\Bigg\}$$

$$- \sum_{j=1}^{M} \Bigg\{ C(2L-j-1,\ M-j)\left[\alpha^{-j}\left(\frac{\mu+s}{\mu-\lambda}\right) + \beta^{-j}\left(1 - \frac{(\mu+s)^2}{(\lambda+s)(\mu-\lambda)}\right)\right]$$

$$- C(2L-j-2,\ M-j)\left[\alpha^{-j-1}\left(\frac{\lambda+s}{\mu-\lambda}\right) - \beta^{-j-1}\left(\frac{\mu+s}{\mu-\lambda}\right)\right]\Bigg\}$$

$$- \frac{(\lambda+\mu+s)}{(\lambda+s)}\left[C(2L-2,\ N-1) + C(2L-2,\ M-1)\right] \Bigg]$$

If $\lambda = \mu$:

$$f_T^*(s) = 2\left(\frac{\lambda}{2\lambda+s}\right)^{2L} \cdot \Bigg[\sum_{j=1}^{L} C(2L-j-1,\ L-1)\alpha^{-j}$$

$$- \sum_{j=1}^{2} \alpha^{-j} \cdot \left[C(2L-j-1,\ N-j) + C(2L-j-1,\ M-j)\right] \Bigg]$$

$$+ \left(\frac{\lambda}{2\lambda+s}\right)^{2L} \cdot \Bigg[\sum_{j=1}^{M} C(2L-j-2,\ M-j)\, j\alpha^{-j-1}$$

$$+ \sum_{j=1}^{N} C(2L-j-2,\ N-j)\, j\alpha^{-j-1}$$

$$+ \sum_{j=3}^{M} C(2L-j-1,\ M-j)\, j\alpha^{-j} +$$

$$+ \sum_{j=3}^{N} C(2L-j-1,\ N-j)\, j\alpha^{-j} \Bigg]$$

For a given number of combatants, L, the effects of changing M and N on total replenishment time can be determined by differentiating $f_T^*(s)$ and setting $s = 0$. This yields the negative of the expected replenishment times for easy comparison.

C. Examples

1. Assume the simple case of two servers and two customers with $M = N = 1$. Then, if $\mu \neq \lambda$,

$$f_T^*(s) = \frac{\lambda^2\mu^2}{(\lambda+\mu+s)^4} \cdot \Bigg\{ \sum_{j=1}^{2} C(3-j,1)\cdot[\alpha^{-j}+\beta^{-j}] - \alpha^{-1}\cdot[C(2,0)+C(2,0)]$$

$$- 2 \sum_{j=1}^{1} \Bigg[C(3-j,1-j)\cdot\Big[\alpha^{-j}\cdot\Big(\frac{\mu+s}{\mu-\lambda}\Big) + \beta^{-j}\cdot\Big(\frac{(\mu+s)^2}{(\lambda+s)(\mu-\lambda)}\Big)\Big]$$

$$- C(2-j,1-j)\cdot\Big[\alpha^{-j-1}\Big(\frac{\lambda+s}{\mu-\lambda}\Big) - \beta^{-j-1}\Big(\frac{\mu+s}{\mu-\lambda}\Big)\Big]\Bigg]\Bigg\}$$

$$= \frac{\lambda^2\mu^2}{(\lambda+\mu+s)^2} \cdot \frac{(\mu+\lambda+2s)^2}{(\lambda+s)^2(\mu+s)^2} = \left[\frac{\mu\lambda(\mu+\lambda+2s)}{(\lambda+\mu+s)(\lambda+s)(\mu+s)}\right]^2$$

Differentiating and setting $s = 0$,

$$E(T) = -f'^*(0) = 2\Big(\frac{1}{\lambda+\mu} - \frac{1}{\lambda} - \frac{1}{\mu}\Big).$$

2. Let $L = 10$ with $M = N = 5$. Then, for $\lambda \neq \mu$,

$$f^*(s) = \frac{(\lambda\mu)^{10}}{(\lambda+\mu+s)^{20}} \cdot \Bigg\{ \sum_{j=1}^{10} C(19-j,9)\cdot(\alpha^{-j}+\beta^{-j}) - 2\alpha^{-1}C(18,4)$$

$$- 2 \sum_{j=1}^{5} \Bigg[C(19-j,5-j)\cdot\Big[\alpha^{-j}\cdot\Big(\frac{\mu+s}{\mu-\lambda}\Big) + \beta^{-j}\cdot\Big(1-\frac{(\mu+s)^2}{(\lambda+s)(\mu-\lambda)}\Big)\Big]$$

$$+ C(18-j,5-j)\Big[\alpha^{-j-1}\cdot\Big(\frac{\lambda+s}{\mu-\lambda}\Big) - \beta^{-j-1}\cdot\Big(\frac{\mu+s}{\mu-\lambda}\Big)\Big]\Bigg]\Bigg\}$$

If $\lambda = 2$ per hour and $\mu = 4$ per hour, after differentiating and setting $s = 0$,

$$E(T) = -f'^*(0) = 5.15 \text{ hours} .$$

BIBLIOGRAPHY

1. Bellman, Richard. "Mathematical Aspects of Scheduling Theory," J. Soc. Indust. Appl. Math, 4: 168 - 205, September, 1956.

2. Gordon, Bradley W., and Copes, Raymond F., III. "An Investigation of Optimal Scheduling of Underway Replenishments," United States Naval Postgraduate School, June, 1967.

3. Hillier, Frederick S., and Boling, Ronald W. "Finite Queues in Series with Exponential or Erlang Service Times--A Numerical Approach," Operations Research, 15: 286 - 303, 1967.

4. Hunt, Gordon C. "Sequential Arrays of Waiting Lines," Operations Research, 4: 674 - 83, 1956.

5. McCullough, David U. "An Application of Queueing Theory to the Operation of Replenishment at Sea," United States Naval Postgraduate School, October, 1966.

6. Sisson, Roger L. "Methods of Sequencing in Job Shops -- A Review," Operations Research, 7: 10 - 29, 1959.

7. Smith, Richard D., and Dudek, Richard A. "A General Algorithm for Solution of the n-Job, M-machine Sequencing Problem of the Flow Shop," Operations Research, 15: 71 - 81, 1967.

APPENDIX

A. Conditional Time Distribution for Steps

The random variables X and Y have exponential distributions with parameters λ and μ, respectively. The conditional distribution of X, given $X < Y$, is developed below:

$$P(X \leq t | X < Y) = 1 - P(X > t | X < Y) = 1 - \frac{P(t < X < Y)}{P(X < Y)}$$

$$P(t < X < Y) = \int_{t < X < Y} \int f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_t^\infty \int_x^\infty \lambda e^{-\lambda x} \cdot \mu e^{-\mu y}\, dy\, dx$$

$$= \int_t^\infty \lambda e^{-(\lambda + \mu)x}\, dx$$

$$= \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}, \quad t \geq 0$$

If $t = 0$, then

$$P(X < Y) = \frac{\lambda}{\lambda + \mu}$$

Therefore, $P(X \leq t | X < Y) = 1 - e^{-(\lambda + \mu)t}$, $t \geq 0$.

B. Counting Techniques for Bounded Step Functions

Step functions having unit steps which are vertically up or horizontally to the right are the only functions permitted in a graph with the following boundaries: $x = M + y$, $x = L$, $y = N + x$, and $y = L$. The boundary $x = M + y$ will be the lower right boundary, and the combined boundaries $x = M + y \leq L$ will be referred to as the right boundary. The upper left boundary is $y = N + x$, and the upper boundary is the combined boundaries $y = x + N \leq L$.

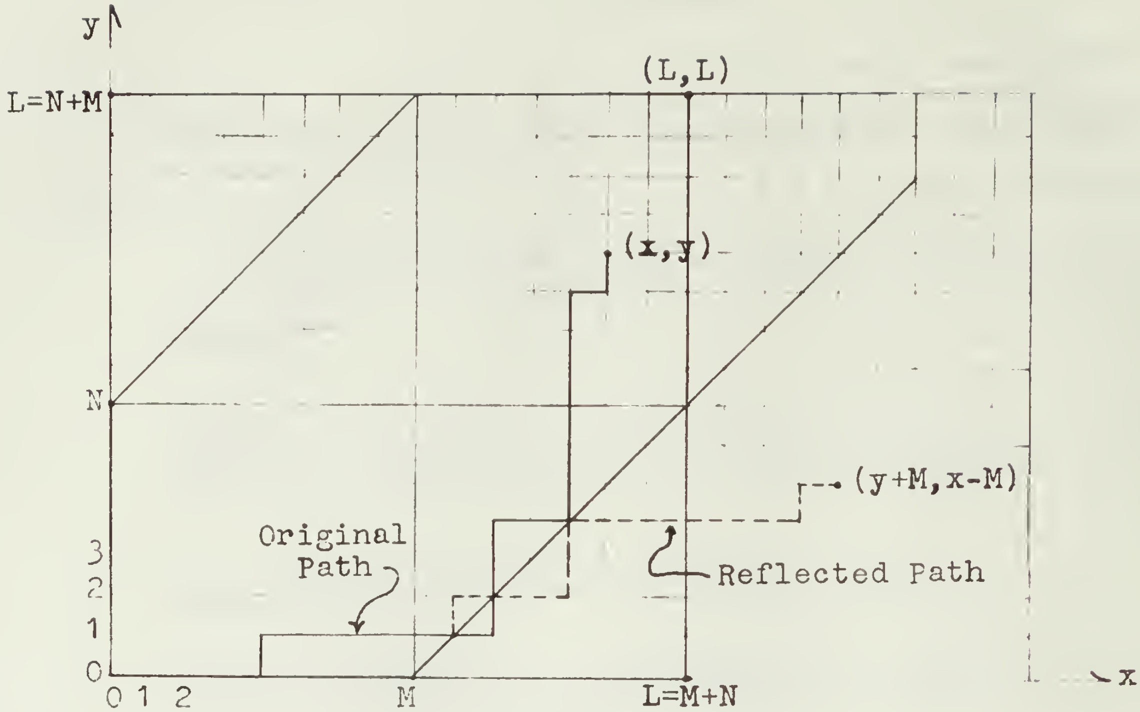

Figure 3

Path illustration of reflection principle about x=y+M.

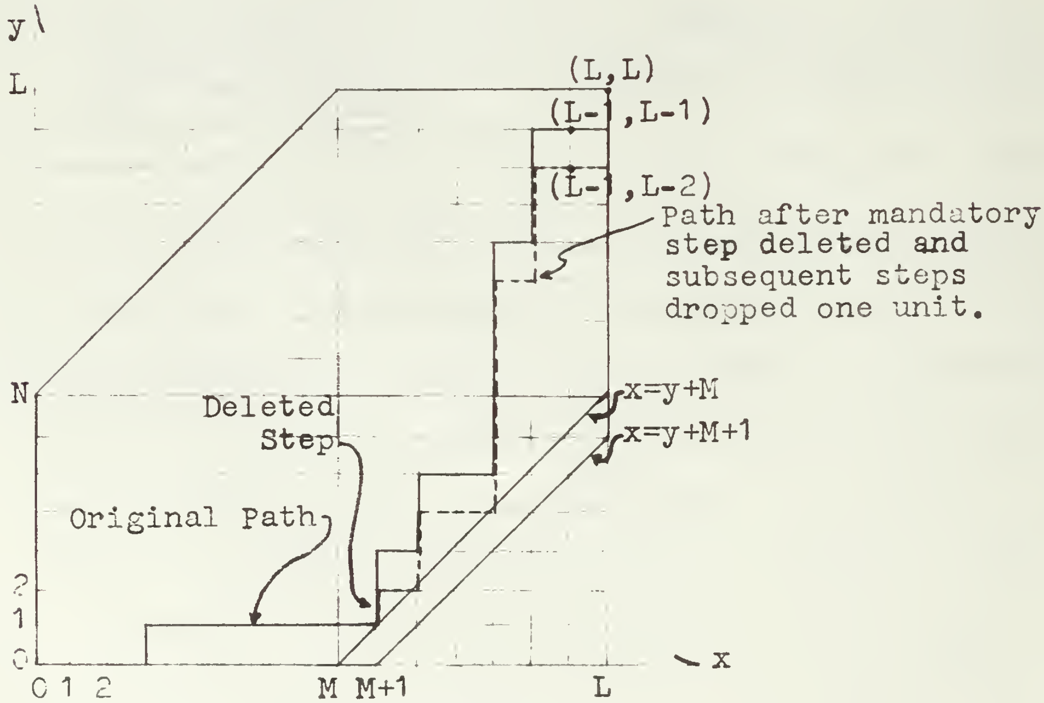

Figure 4

Illustration of step function that hits lower right boundary once.

First a technique for counting the number of step functions from $(0,0)$ to (x,y), $M \leq x < L$ and $N \leq y < L$, that do not touch the upper left or lower right boundaries must be developed. To do this, the number of paths or routes (both terms used interchangeably with step functions) that have a point in common with these boundaries will be subtracted from the number of routes that are possible without considering any boundaries.

The number of steps required to go from $(0,0)$ to (x,y) by any route is x steps to the right and y steps up. The number of possible combinations of $x + y$ steps with x steps to the right is $C(x + y, x) = (x + y)!/x!y!$. This is also the number of routes that are possible without considering any boundaries.

Next, consider a path that hits the lower right boundary of the graph as exemplified in figure 3. If the part of this route from the first point of contact with the boundary to (x,y) is reflected symmetrically about the boundary, then the reflected route (dotted line in diagram) has the same number of steps to the point $(y + M, x - M)$ as does the original step function to (x,y). Also, if any path that proceeds to $(y + M, x + N)$ is reflected symmetrically from the first point of contact with $x = y + M$, the reflected route proceeds to (x,y). Therefore, the two sets of routes are equivalent, and it is only necessary to count the step functions from $(0,0)$ to $(y + M, x - M)$ in order to determine the number of routes touching the lower right boundary. This number is $C(x+y,y+M)$.

Similarly, by using the reflection principle about $y = x + n$, the number of routes touching the upper left boundary is $C(x + y, x + N)$. Therefore, the total possible number routes from (0,0) to (x,y), $M \leqq x < L$, $N \leqq y < L$, that do not touch either of the boundaries is

$$C(x + y, x) - C(x + y, y + M) - C(x + y, x + N).$$

Note that this relation is dependent on M and N, the intersection of the boundaries with the axes, so that the boundaries may be shifted and the total number of routes not touching the new boundaries may be determined by the above method.

It can be shown that, in general, the number of possible routes from any point (x_o, y_o) to(x,y) that do not touch the boundaries $x = y + M$ and $y = x + N$ is

$$C(x-x_o+y-y_o,x-x_o) - C(x-x_o+y-y_o,y-x_o+M) - C(x-x_o+y-y_o,x-y_o+N).$$

Now define R_{ij} as the number of possible paths from (0,0) to (L,L) that touch positions on the upper boundary i times and the right boundary j times. Since a diagonal step is not permitted, every path must pass through (L - 1, L) or (L, L - 1) to get to (L, L), and so R_{00} is obviously zero. Consider R_{0j} next, where $j = 1, 2, \ldots, L$. In this case, no routes hit the upper boundary.

To formulate a method for finding R_{0j} in general, it will be shown that a set A, consisting of routes hitting positions on the right boundary exactly j times, is equivalent to a set B, consisting of routes that proceed to

$(L - 1, L - j)$ without hitting the boundaries $x = y + m + j - 1 \leqq L$ and $y = x + N \leqq L$. This being true, the technique developed to count routes within specific boundaries can be applied directly.

If $j = 1$, set A paths pass through $(L - 1, L - 1)$ without hitting either of the original boundaries and then move to the right. Set B paths also proceed to $(L - 1, L - 1)$ without hitting the boundaries $x = y + M \leqq L$ and $y = x + N \leqq L$. The two sets are obviously equivalent because they are the same sets, and

$$R_{01} = C(L-1+L-1, L-1) - C(2L-2, L-1-N) - C(2L-2, L-1-M).$$

If j is greater than one, however, the equivalence is not so obvious because j different types of paths are possible. For descriptive ease, "type K" will indicate the type of path that kits K positions on the boundary $x = L$ and $j - K$ positions on the lower right boundary. For example, let $j = 2$, and consider set A. Two types of routes are possible, type I and type II. A type II route passes through $(L - 1, L - 2)$ without hitting any boundary and proceeds to $(L, L - 2)$. A type I route hits the lower right boundary exactly once, proceeds to $(L - 1, L - 1)$, and then to the right.

For $j = 2$, since all routes in set B proceed to $(L-1, L-2)$, type II routes in set A correspond to the same routes in set B. A type I route, however, hits the lower right boundary once, and the next step after hitting the boundary must be up. If this mandatory step up is deleted from consideration, and all subsequent steps of the path are dropped

one unit down, as indicated by the dotted path in figure 4, the path proceeds to $(L - 1, L - 2)$ and then right, never touching $x = y + M + 1$. Therefore, all routes in set A have corresponding routes in B.

In set B, when $j = 2$, all routes proceed to $(L - 1, L - 2)$ and never touch $x = y + M + 1$. Consider a route that hits $x = y + M$. If the route is raised one step after hitting this line for the first time by inserting an extra vertical step, then the route proceeds to $(L - 1, L - 1)$ and corresponds exactly to a type I route in set A. The remaining routes in B do not hit either $x = y + M + 1$ or $x = y + M$ and still proceed to $(L - 1, L - 2)$. These routes correspond to identical type II routes in A. Consequently, the two sets are equivalent, and, using set B,

$$R_{02} = C(L-1+L-2, L-1) - C(2L-3, L-2+M+1) - C(2L-3, L-1+N)$$
$$= C(2L-3, L-1) - C(2L-3, N-2) - C(2L-3, M-2).$$

In general, when $i = 0$, j types of routes are possible among the R_{0j} routes in set A: types I, II, ..., and j. A type $j - 1$ route hits the lower right boundary one time and then proceeds to $(L-1, L-j-1)$. If the mandatory step up after hitting the boundary is deleted and the remaining part of the route is dropped one unit, the path will proceed to $(L-1, L-j)$ without hitting $x = y + M + 1$. For a type K route, $I \leq K \leq j$, $j - K$ steps must be up when the lower right boundary is hit. If these mandatory steps up are deleted and the remainder of the route is dropped one unit following each deleted step, the path proceeds to $(L-1, L-j)$ and corresponds

to a route in B that does not hit $x = y + M + K$. Therefore, all routes in A correspond to routes in B.

In set B, consider all routes that do not hit $x = y + M$. These routes correspond exactly to the type j routes in set A. Next, consider a route in B that hits $x = y + M + j-1-K$ and does not hit $x = y + M + j - K$, $I \leq K \leq j - 1$. If a vertical step is added to each route immediately following the first and subsequent times the boundary $x = y + M$ is hit, and the remainder of the route is raised one unit for each step added, the raised route will correspond to type K route in set A. Therefore, all routes in B correspond to routes in A, and set A is equivalent to set B. Consequently, for any j,

$$R_{0j} = C(L-1+L-j, L-1) - C(2L-1-j, L-j+M+j-1) - C(2L-1-j, L-1+N)$$
$$= C(2L-1-j, L-1) - C(2L-1-j, N-j) - C(2L-1-j, M-j).$$

By interchanging i and j, M and N, it is obvious that

$$R_{i0} = C(2L-1-i, L-1) - C(2L-1-i, M-i) - C(2L-1-i, N-i).$$

Consider now R_{ij} when $i \neq 0$ and $j \neq 0$. Define R'_{ij} as the number of paths that hit the lower right boundary first and then proceed to (L-i, L), and R''_{ij} as the number of paths that hit the upper left boundary first and proceed to (L, L-j). Then, if $i \neq 0$ and $j \neq 0$, $R_{ij} = R'_{ij} + R''_{ij}$.

Let $i = 1$. Then, for R'_{ij}, the number of routes that hit the lower right boundary j times and proceed to (L-1, L-1) must be found. Consider type I routes among the set A routes that hit the right boundary $j + 1$ times and do not hit the upper boundary. These routes hit the lower right boundary j times and proceed to (L-1, L-1), which is exactly what

is being sought. Therefore, for R'_{ij}, it is necessary only to find the number of type I routes for set A. Considering the previous set B, this is simply $R_{0,j+1}$ less all those routes that do not hit the boundaries $x = y + M + (j + 1) - 2 = y + M - j - 1$ and $y = x + N$ when proceeding to $(L-1, L-j-1)$. Therefore,

$$R'_{ij} = [C(2L-1-(j+1),L-1) - C(2L-j-2,N-j-1) - C(2L-j-2,M-j-1)]$$
$$- [C(L-1+L-j-1,L-1) - C(2L-j-2,L-j-1+M+j-1)-C(2L-j-2,L-1-N)]$$
$$= C(2L-j-2,N-j) - C(2L-j-2,N-j-1).$$

Let i be greater than one, and consider a new right boundary, $x = y + M \leqq L - i + 1$. If the number of type I routes hitting this boundary j+1 times and not hitting $y = x + N$ is found, then R'_{ij} can be determined. By the same techniques used to formulate R_{0j} when $x = y + M \leqq L$, if the right boundary is $x = y + M \leqq L + 1 - i$, then

$$R^{*}_{0,j+1} = C(L-i+L-(j+1),\ L-i) - C(2L-i-j-1,\ L-j-1+M+(j+1)-1)$$
$$- C(2L-i-j-1,\ L-i+N)$$
$$= C(2L-i-j-1,\ L-i) - C(2L-i-j-1,\ N-i-j)$$
$$- C(2L-i-j-1.\ M-j-1).$$

The number of type I routes for this case is, therefore, equal to $R^{*}_{0,j+1}$ less the number of routes proceeding to $(L-i, L-(j+1))$ without touching the boundaries $x = y + M + (j+1) - 1$ and $y = x + N$. Thus, for $i \neq 0$, $j \neq 0$,

$$R'_{ij} = [C(2L-i-j-1,\ L-i) - C(2L-i-j-1,\ N-i-j) - C(2L-i-j-1,$$
$$M-j-1)] - [C(2L-i-j-1,\ L-i) - C(2L-i-j-1,\ L-j-1+M+j-1)$$
$$- C(2L-i-j-1,\ L-i-N)]$$
$$= C(2L-i-j-1,\ N-i-j+1) - C(2L-i-j-1,\ N-i-j).$$

If $i + j = k$, then for $i \neq 0$ and $j \neq 0$,

$$R'_{ij} = C(2L-k-1,\ N-k+1) - C(2L-k-1,\ N-k).$$

Similarly, by interchanging M and N and i and j,

$$R''_{ij} = C(2L-k-1,\ M-k+1) - C(2L-k-1,\ M-k).$$

In general, therefore, with $i + j = k$, $i \neq 0$, $j \neq 0$,

$$R_{ij} = C(2L-k-1,\ N-k-1) + C(2L-k-1,\ M-k-1) - C(2L-k-1,\ N-k) - C(2L-k-1,\ M-k).$$

It should be noted that, although the maximum ranges of i and j are from 0 to L, the equations hold for all values of i and j; since, if i or j is greater than L, the combinatorial forms will be zero.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
2.	Library Naval Postgraduate School Monterey, California	2
3.	Chief of Naval Operations (OP-96) Department of the Navy Washington, D. C. 20350	1
4.	Professor Paul R. Milch (Thesis Advisor) Operations Analysis Department Naval Postgraduate School Monterey, California	1
5.	Professor Stephen M. Pollock Operations Analysis Department Naval Postgraduate School Monterey, California	1
6.	Lt. Mark H. Waggoner 322 West Henrietta Kingsville, Texas 78363	1
7.	Howard G. Bergman Replenishment-at-Sea System Project Office (PMS-90) Naval Ship Systems Command Washington, D. C. 20360	1

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

AN ANALYTICAL MODEL FOR APPLICATION TO THE OPERATION OF REPLENISHMENT AT SEA

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Thesis

5. AUTHOR(S) *(Last name, first name, initial)*

WAGGONER, MARK HARVEY, Lieutenant, USN

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1967	37	7

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. AVAILABILITY/LIMITATION NOTICES

~~This document is subject to special export controls and each transmittal to foreign nationals may be made only with prior approval of the Naval Postgraduate School.~~

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California

13. ABSTRACT

An analytical model for replenishment at sea is formulated for two supply ships and L combatants using queueing theory concepts and a random walk model in the plane. Exponential distributions are assumed for replenishment times, and, given the initial number of combatants to be replenished by each supply ship, the distribution for total time to complete the finite operation is obtained in terms of Laplace transforms. All possible sequences for finishing the replenishments of the combatants have been considered in the model, and the techniques which were developed to count the number of sequence possibilities are presented as an appendix. Although this model involves only two supply ships, it is believed that the methods used may be generalized for application to more complicated models.

UNCLASSIFIED
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Replenishment at Sea						
Analytical Model						
Random Walk in Plane						
Queueing Theory						
Underway Replenishment						

DD FORM 1 NOV 65 1473 (BACK)

S/N 0101-807-6821

UNCLASSIFIED
Security Classification